

Fast and simple decycling and dismantling of networks

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Decycling and dismantling of complex networks are underlying many important applications in network science. Recently these two closely related problems were tackled by several heuristic algorithms, simple and considerably sub-optimal, on the one hand, and time-consuming message-passing ones that evaluate single-node marginal probabilities, on the other hand. In this paper we propose a simple and extremely fast algorithm, CoreHD, which recursively removes nodes of the highest degree from the 2-core of the network. CoreHD performs much better than all existing simple algorithms. When applied on real-world networks, it achieves equally good solutions as those obtained by the state-of-art iterative message-passing algorithms at greatly reduced computational cost, suggesting that CoreHD should be the algorithm of choice for many practical purposes.

I. INTRODUCTION

In decycling of a network we aim to remove as few nodes as possible such that after the removal the remaining network contains no loop. In network dismantling we aim to find the smallest set of nodes such that after their removal the network is broken into connected components of sub-extensive size. These are two fundamental network-optimization problems with a wide range of applications, related to optimal vaccination and surveillance, information spreading, viral marketing, and identification of influential nodes. Considerable research efforts have been devoted to the network decycling and dismantling problems recently [1–8].

Both the decycling and the dismantling problems belong to the class of NP-hard problems [6, 9], meaning that it is rather hopeless to look for algorithms to solve them exactly in polynomial time. However, finding the best possible approximate solutions for as large classes of networks as possible is an open and actively investigated direction. Recent theoretic and algorithmic progress on both these problems [1–3, 5, 6] came from the fact that, on random sparse networks with degree distributions having a finite second moment, methods from physics of spin glasses provide accurate algorithms for both decycling and dismantling. These sparse random networks are locally tree-like and do not contain many short loops. On such networks the decycling is closely linked to dismantling and asymptotically almost the same set of nodes is needed to achieve both [5, 6, 10]. Even on real-world networks that typically contain many small loops, best dismantling is currently achieved by first finding a decycling solution and then re-inserting nodes that close short loops but do not increase too much the size of the largest component [5, 6].

Both the algorithms of [6] and [5] achieve performance that is extremely close to the theoretically optimal values computed on random networks. However, both these algorithms are global, they need to iterate certain equa-

tions on the whole network in order to select the suitable candidate nodes. Although they are both scalable and can be run on networks with many millions of nodes, they are not completely straightforward to understand and require some experience with spin glass theory. The close-to-optimal performance of these algorithms is theoretically justified only on random networks. Despite their good performance observed empirically on networks with many loops, there might still exist even better and analyzable strategies for real-world networks.

As usual in applied science, in many potential applications we are at first not even sure that optimal dismantling or optimal decycling is the best strategy to answer the question in hand (e.g., the problem of social influence maximization [11–14]). Therefore it is extremely important to have a really very simple and fast decycling and dismantling strategy that can provide an accurate assessment of whether this approach is at all interesting for a given practical problem. However, existing simple strategies, such as removing adaptively high degree nodes [15, 16], are very far from optimal performance and therefore not very useful. Recently the authors of [4] claimed that a heuristics based on the so-called *collective influence* (CI) measure can be a perfect candidate for this purpose. This algorithm has attracted a lot of enthusiasm in the network science community. However, more systematic investigations performed in [5, 6, 8] revealed that the CI algorithm is still annoyingly far from being optimal. The CI algorithm is also not particularly competitive in terms of computational time because a large neighborhood of a node needs to be considered in order to evaluate the CI measure.

In the present paper we introduce the CoreHD algorithm that is basically as simple and fast as the adaptive removal of high degree nodes, yet its performance is much closer to optimal than the CI algorithm or its extended versions, and comparably close as the best known message-passing methods [5, 6] while several orders of magnitude faster. It hence provides simple and tractable

solutions for networks with many billions of nodes. The method is simply based on adaptive removal of highest-degree nodes from the 2-core of the network. Apart of its simplicity and speed the performance of the CoreHD algorithm is basically indistinguishable from the performance of the message-passing algorithms on random graphs with long-tailed degree distributions. On all real-world network instances we tested the result by CoreHD is within few nodes from the best one found by message-passing and on some instances we found that it is even slightly better. On top of all that, the simple structure of CoreHD might be amenable to rigorous analysis providing guarantees for loopy networks that are not accessible for the message-passing methods.

For all the above reasons we argue that in many applications of decycling and dismantling CoreHD should be the first choice. The simple algorithmic idea generalizes easily to the problem of destroying optimally the k -core of a network - one focuses on the current k -core and adaptively removes highest degree nodes.

II. THE COREHD ALGORITHM

We now describe CoreHD as an extremely fast algorithm for decycling and dismantling of huge complex networks with close-to-optimal outcomes. Let us begin with some motivating discussions.

Perhaps the simplest algorithms one can propose for decycling and dismantling is adaptive removal of highest-degree nodes. We call this method HD, it is indeed extremely fast, but empirically does not perform very well. One reason why HD does not work well is that some nodes of large degree, such as node i in Fig. 1, do not belong to any loop, and hence do not have to be removed for decycling. Due to the property that trees can always be dismantled by a vanishing fraction of nodes [10], nodes such as i of Fig. 1 are also not useful for dismantling. Note that the CI method of [4] shares this problem, see the appendix.

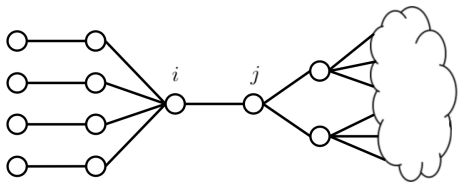


FIG. 1. Illustration of a network with dangling trees. Each circle denotes a node in the network, each line connecting circles denotes an edge, and the cloud represents the other part (nodes and edges) of the network.

The above observation motivates a very natural idea that dismantling and decycling algorithms should always focus only on the 2-core of the network. The 2-core is a sub-network that is obtained after adaptive removal of all leaves (nodes with only a single attached edge). The sim-

plest and fastest strategy is then to remove the highest-degree nodes from the remaining 2-core. To our surprise this simple idea provides much better performance than other comparably simple approaches existing in the literature. We call the resulting algorithm CoreHD, it is detailed in Algorithm 1.

Algorithm 1: CoreHD

Input: A network.

Output: A forest of small trees.

1. Find the 2-core of the network, and obtain the degree of every node within this 2-core (edges to outside nodes not considered).
 2. Identify the node i with the largest degree in the 2-core. If there are more nodes with the same largest degree, randomly choose one of them.
 3. Remove node i , update the 2-core and the degrees of all its nodes. If the 2-core is empty, then do tree-breaking and stop; otherwise go to step 2.
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For the decycling problem, CoreHD simply removes highest-degrees nodes from the 2-core in an adaptive way (updating node degree as the 2-core shrinks), until the remaining network becomes a forest. For dismantling, after decycling, CoreHD also breaks the trees into small components, see Appendix that follows tree-breaking strategy from [5, 6]. In case the original network has many small loops, a refined dismantling set is obtained after a reinsertion of nodes that do not increase (much) the size of the largest component, again as proposed recently in [5, 6]. For details on implementation of the reinsertion algorithm we refer to the Appendix.

III. RESULTS

In this section we evaluate the CoreHD algorithm for both random and real-world networks, by comparing the minimum fraction of nodes we need to remove in order to break the network into a forest or components with size smaller than $0.01n$. We compare to the Belief Propagation guided Decimation (BPD) [5] and Collective Influence method (CI) [4] (CI₄ results are obtained using the original code of Ref. [4]).

First, we notice that on some simple examples, e.g. regular random graphs with degree 3, the CoreHD algorithm reaches the exact optimal decycling fraction $\rho = 0.25$. This matches the performance of a greedy method of [17] that for this particular case is provably optimal.

In Fig. 2 we compare the performance of the above algorithms on an Erdős-Rényi random network with $N = 50000$ nodes and average degree $c = 3.5$. In the upper panel we plot the fraction of nodes in the largest connected component (LCC, denoted q) as a function of the fraction of removed nodes, denoted ρ . We see that compared to HD and CI the CoreHD algorithm works

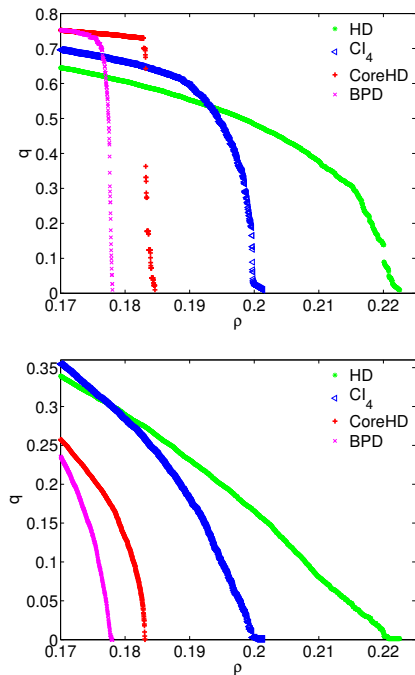


FIG. 2. Fraction of nodes in the largest connected component (LCC) (*upper*) and in the 2-core (*lower*) as a function of fraction of nodes removed, for HD, CI₄, CoreHD and BPD on an Erdős-Rényi random graph with number of nodes $N = 5 \times 10^4$, and average degree $c = 3.5$. In all four methods nodes are removed one by one.

the best by a large margin, breaking the network into small component with size smaller than $0.01N$ after removing fraction of only 0.1846 of nodes. While CI and HD need to remove fraction 0.2014, and 0.2225 of nodes respectively. This is compared to the close-to-optimal performance of the iterative message passing BPD that needs to remove fraction 0.1780 of nodes, and to the theoretical prediction for the asymptotically optimal value 0.1753 [1–3, 6, 7].

We also see from the figure that the fraction of nodes in the LCC obtained by CoreHD encounters a first order transition when $\rho_{\text{dec}} = 0.1831$, this is because at this point (just at the beginning of the discontinuity) the remaining network becomes a forest. The greedy tree-breaking procedure then quickly breaks the forest into small components. While the other algorithms do not have this phenomenon, the size of the LCC goes to zero continuously. In the lower panel of Fig. 2 we plot the fraction q of nodes in the 2-core as a function of ρ . We can see that for CoreHD, q reaches zero at $\rho = 0.1831$ indicating that the remaining network contains no loop, thus is a forest. While for other algorithms the 2-core remains extensive until the network is dismantled. On a larger ER random network with $N = 10^6$, $c = 3.5$, the difference between the sizes of decycling and the dismantling sets the CoreHD algorithm finds is not distinguishable within the precision of 4 valid digits and is 0.1830 for

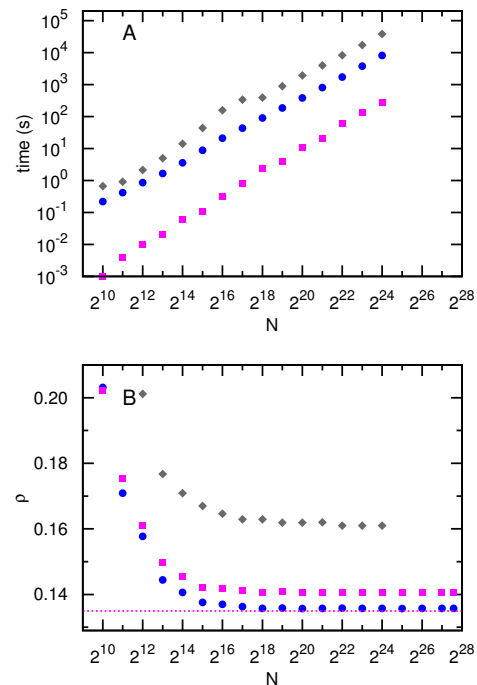


FIG. 3. Performance of the CoreHD algorithm (magenta squares) and its comparison with the BPD algorithm (blue circles) and the CI algorithm ($\ell = 4$, grey diamonds) on ER networks of average degree $c = 3$ and size N . (A) The relationship between the total running time τ and N . The simulation results are obtained on a relatively old desktop computer (Intel-6300, 1.86 GHz, 2 GB memory). (B) The relationship between the fraction ρ of removed nodes and N . The dotted horizontal line denotes the theoretically predicted minimum value.

both. Note that this result is (slightly) better than yet another approach suggested recently in the literature [8] that achieves 0.1838 with an algorithm still considerably more involved than CoreHD.

Besides performing much better than CI, the CoreHD is also much faster: the 2-core of the network can be computed efficiently using a leaf-removal process with $O(N)$ operations. After deleting a node, one only needs to update the 2-core, which requires on average $O(1)$ operations in sparse networks, and is clearly much faster than updating the CI score. Actually, in sparse networks when the size of the 2-core is much smaller than the size of the network, CoreHD is even faster than the HD algorithm which removes one by one nodes from the whole network.

The computational times for the CoreHG, CI and BPD algorithms as the system size grows are shown in Fig. 3 for ER network with mean degree $c = 3$. The BPD algorithm performs slightly better than the CoreHD algorithm but it is much slower. For example, for an ER network with $c = 3$ and $N = 2 \times 10^8$, the solution obtained by CoreHD has relative dismantling/decycling set size $\rho \approx 0.1407$ (computing time is 64 minutes), which is only slightly larger than the value of $\rho \approx 0.1357$ ob-

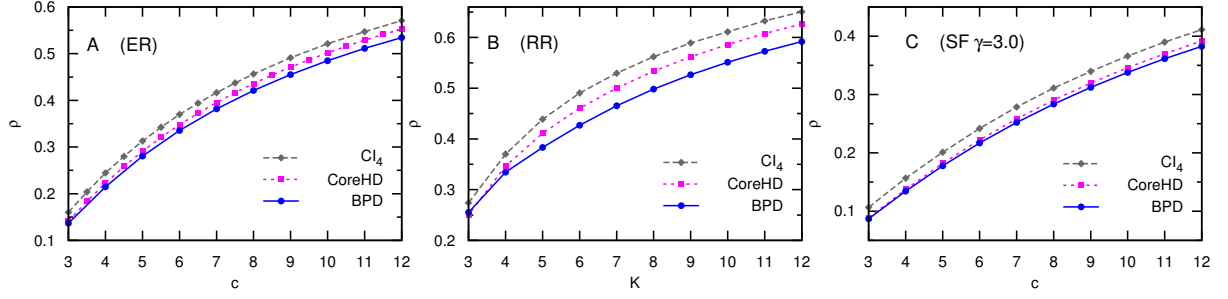


FIG. 4. Fraction ρ of removed nodes for (A) Erdős-Rényi (ER) random networks of mean degree c , (B) Regular Random (RR) networks of degree K , and (C) Scale Free (SF) networks of average degree c and decay exponent $\gamma = 3.0$ generated as in [18]. Each data point obtained by CoreHD is the over 96 instances of size $N = 10^5$. The results of CI_4 and the results of BPD are from [5]. In BPD and CI_4 , at each iteration a fraction f of nodes are removed (with $f = 0.01$ for BPD and $f = 0.001$ for CI_4 , decreasing f does not improve the performance visibly), while in CoreHD nodes are removed one by one.

TABLE I. Comparative results of the CoreHD method with CI and the BPD algorithm on a set of real-world network instances. N and M are the number of nodes and links of each network, respectively. The number of nodes deleted by CI, CoreHD, and BPD are listed in the 4th, 5th, and 6th column. The CI and BPD results are from [5]. The time (seconds) for dismantling is the running time of algorithms, i.e. with time for reading network from the data file excluded.

Network	N	M	decycling		dismantling			Time for dismantling		
			CoreHD	BPD	CI	CoreHD	BPD	CI	CoreHD	BPD
RoadEU [19]	1177	1417	90	91	209	148	152	0.18	< 0.001	0.1
PPI [20]	2361	6646	365	362	424	357	350	0.91	< 0.001	2.09
Grid [21]	4941	6594	519	512	476	327	320	1.00	< 0.001	0.66
IntNet1 [22]	6474	12572	217	215	198	156	161	5.19	< 0.001	11.32
Authors [23]	23133	93439	8311	8317	3588	2600	2583	87.55	0.09	40.04
Citation [22]	34546	420877	15489	15390	14518	13523	13454	4166	0.2	383.91
P2P [24]	62586	147892	9557	9285	10726	9561	9292	520.59	0.21	50.24
Friend [25]	196591	950327	38911	38831	32340	27148	26696	5361	1.37	588.19
Email [23]	265214	364481	1189	1186	21465	1070	1064	6678	0.39	151.57
WebPage [26]	875713	4322051	208509	208641	106750	51603	50878	2275	9.67	2532
RoadTX [26]	1379917	1921660	243969	239885	133763	20289	20676	273.69	4.07	421.15
IntNet2 [22]	1696415	11095298	229034	228720	144160	73601	73229	19715	35.84	4243

tained by BPD (computing time is 23.5 hours [5]). We note that in these experiments, in each step of removal, BPD and CI_4 remove 0.1% of nodes (e.g., 10000 nodes for $N = 10^7$), while CoreHD removes only 1 node per step. Even this way the computational time of CoreHD is shorter than the time used for reading the network from the data file (edge-list format, using a c++ procedure).

Fig. 4 presents results for Erdős-Rényi random graphs, regular random graphs, and scale-free random networks of varying average degree. In all cases CoreHD works better than CI_4 and worse than BPD, with the best performance obtained for scale-free networks. The good performance of CoreHD for the scale-free networks is of particular interest because almost all real-world networks have a heavy-tailed degree-distribution.

A set of experiments on real-world networks is presented in Tab. I. We list the fraction of nodes we need to remove in order to remove all cycles, and in order to break the network into small components with size smaller than $0.01N$. For dismantling, in addition to Al-

gorithm 1 we do a refinement by inserting back some deleted nodes that do not increase the largest component size beyond the $0.01N$. We can see that CoreHD works excellently for real-world network instances, giving decycling and dismantling sets very close to the state-of-art BPD and much smaller than CI_4 . It is also surprising to see that in some networks e.g. RoadEU, IntNet1 and RoadTX, CoreHD even outperforms BPD slightly. Tab. I clearly demonstrates the time superiority of CoreHD for real-world networks as compared with both CI_4 and BPD.

IV. CONCLUSION AND DISCUSSIONS

We have presented that iteratively removing nodes having the highest degree from the 2-core of a network gives an ultra-fast while very efficient algorithm for decycling and dismantling of networks. Our algorithm is so fast that its running time is shorter than the time of reading the network file.

It is still surprising to us that such a simple algorithm could work much better than more sophisticated algorithms: We have tried running CI (see Appendix), adjacency matrix centrality on the 2-core of the network, and HD on 3-core of the network, they are all slower but perform no better than CoreHD. Our experiments also show that CoreHD outperforms centrality measures using left and right eigenvector of the non-backtracking matrix [27], an idea that originally inspired us to propose the CoreHD algorithm. More detailed understanding of why this is the best performing strategy is left for future work.

On the real-world networks which typically have many short loops and motifs, decycling is quite different from dismantling. A natural idea to generalize our CoreHD would be consider a factor graph treating short loops

and motifs as factors, then do CoreHD on the 2-core of the factor graph.

Finally, CoreHD can be generalized naturally to removal of the k -core, again running the adaptive DH heuristics on the k -core or the current graph. Comparison of this strategy to existing algorithms [2, 28] is in progress.

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- [1] H.-J. Zhou. Spin glass approach to the feedback vertex set problem. *Eur. Phys. J. B*, 86:455, 2013.
 - [2] F. Altarelli, A. Braunstein, L. Dall'Asta, and R. Zecchina. Optimizing spread dynamics on graphs by message passing. *Journal of Statistical Mechanics: Theory and Experiment*, 2013(09):P09011, 2013.
 - [3] A. Guggiola and G. Semerjian. Minimal contagious sets in random regular graphs. *Journal of Statistical Physics*, 158(2):300–358, 2015.
 - [4] F. Morone and H. A. Makse. Influence maximization in complex networks through optimal percolation. *Nature*, 524:65–68, 2015.
 - [5] S. Mugisha and H.-J. Zhou. Identifying optimal targets of network attack by belief propagation. *arXiv preprint arXiv:1603.05781*, 2016.
 - [6] A. Braunstein, L. Dall'Asta, G. Semerjian, and L. Zdeborová. Network dismantling. *arXiv preprint arXiv:1603.08883*, 2016.
 - [7] S.-M. Qin, Y. Zeng, and H.-J. Zhou. Spin glass phase transitions in the random feedback vertex set problem. *arXiv preprint arXiv:1603.09032*, 2016.
 - [8] P. Clusella, P. Grassberger, F. J. Pérez-Reche, and A. Politi. Immunization and targeted destruction of networks using explosive percolation. *arXiv:1604.00073*, 2016.
 - [9] R. M. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer, 1972.
 - [10] S. Janson and A. Thomason. Dismantling sparse random graphs. *Combinatorics, Probability and Computing*, 17(02):259–264, 2008.
 - [11] M. Richardson and P. Domingos. Mining knowledge-sharing sites for viral marketing. In *Proceedings of 8th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 61–70, New York, NY, 2002. ACM.
 - [12] D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. *Theory of Computing*, 11:105–147, 2015.
 - [13] K. Jung, W. Heo, and W. Chen. Irie: Scalable and robust influence maximization in social networks. In *2012 IEEE 12th International Conference on Data Mining*, pages 918–923. IEEE, 2012.
 - [14] C. Borgs, M. Brautbar, J. Chayes, and B. Lucier. Maximizing social influence in nearly optimal time. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 946–957. Society for Industrial and Applied Mathematics, 2014.
 - [15] R. Albert, H. Jeong, and A.-L. Barabási. Error and attack tolerance of complex networks. *nature*, 406(6794):378–382, 2000.
 - [16] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin. Breakdown of the internet under intentional attack. *Physical review letters*, 86(16):3682, 2001.
 - [17] S. Bau, N. C. Wormald, and S. Zhou. Decycling numbers of random regular graphs. *Random Structures & Algorithms*, 21(3-4):397–413, 2002.
 - [18] K.-I. Goh, B. Kahng, and D. Kim. Universal behavior of load distribution in scale-free networks. *Phys. Rev. Lett.*, 87:278701, 2001.
 - [19] L. Šubelj and M. Bajec. Robust network community detection using balanced propagation. *Eur. Phys. J. B*, 81:353–362, 2011.
 - [20] D. Bu, Y. Zhao, L. Cai, H. Xue, X. Zhu, H. Lu, J. Zhang, S. Sun, L. Ling, N. Zhang, G. Li, and R. Chen. Topological structure analysis of the protein-protein interaction network in budding yeast. *Nucleic Acids Res.*, 31:2443–2450, 2003.
 - [21] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–442, 1998.
 - [22] J. Leskovec, J. Kleinberg, and C. Faloutsos. Graphs over time: densification laws, shrinking diameters and possible explanations. In *Proceedings of the eleventh ACM SIGKDD international conference on Knowledge discovery in data mining*, pages 177–187. ACM, New York, 2005.
 - [23] J. Leskovec, J. Kleinberg, and C. Faloutsos. Graph evolution: Densification and shrinking diameters. *ACM Transactions on Knowledge Discovery from Data*, 1:2, 2007.
 - [24] M. Ripeanu, I. Foster, and A. Iamnitchi. Mapping the gnutella network: Properties of large-scale peer-to-peer systems and implications for system design. *IEEE Inter-*

- net Comput.*, 6:50–57, 2002.
- [25] E. Cho, S. A. Myers, and J. Leskovec. Friendship and mobility: User movement in location-based social networks. In *ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 1082–1090, San Diego, CA, USA, 2011.
 - [26] J. Leskovec, K. J. Lang, A. Dasgupta, and M. W. Mahoney. Community structure in large networks: Natural cluster sizes and the absence of large well-defined clusters. *Internet Math.*, 6:29–123, 2009.
 - [27] P. Zhang. Nonbacktracking operator for the ising model and its applications in systems with multiple states. *Phys. Rev. E*, 91:042120, Apr 2015.
 - [28] S. Pei, X. Teng, J. Shaman, F. Morone, and H. A. Makse. Collective influence maximization in threshold models of information cascading with first-order transitions. *arXiv preprint arXiv:1606.02739*, 2016.
 - [29] F. Krzakala, C. Moore, E. Mossel, J. Neeman, A. Sly, L. Zdeborová, and P. Zhang. Spectral redemption in clustering sparse networks. *Proc. Natl. Acad. Sci. USA*, 110(52):20935–20940, 2013.

Appendix A: Greedy Tree Breaking and Refinement by Insertion

Optimally breaking a forest into small components can be solved in polynomial time [10]. Empirically a greedy tree-breaking procedure works very well. In such a greedy dynamics we iteratively find and remove the node which leads to the largest drop in the size of the largest connected component.

In more details, the largest component caused by removal of each node in a tree can be computed iteratively (see, e.g. [5, 6]). Starting from a leaf, each node sends a message to each of its neighbors, reporting the largest component caused by removing the edge between them. After the messages arrive at the root of the tree, we can then easily identify the node such that its removal decreases maximally the component size.

For the refinement, we also use a simple greedy strategy to insert back some of the removed nodes [5, 6]. In each step of re-insertion, we calculate the increase of the component size after the insertion of a node, and then identify the node which gives the smallest increase.

Appendix B: Dangling-tree problem of the CI index

The collective influence index was proposed in [4] as a measure of node’s importance in influence spreading. At a given level ℓ the CI index of a node i is defined as

$$CI_\ell(i) = (d_i - 1) \sum_{j \in \partial_i^\ell} (d_j - 1), \quad (B1)$$

where d_i is the degree of node i in the remaining network, and ∂_i^ℓ denotes the set of nodes that are at distance ℓ from node i . In the CI algorithm, a small fraction f (e.g., $f = 0.001$) of nodes with the highest CI values are removed from the network and then the CI indices of the remaining nodes are updated. The authors of [4] claimed that the $CI_\ell(i)$ approximates the eigenvector of the non-backtracking operator [29].

However we can see immediately that CI has a drawback which does not reflect the functioning of the non-backtracking operator. We illustrate this in an example network shown in Fig. 1. Without loss of generality let us consider $\ell = 2$, then it is easy to see that the node i of this figure has $CI_2(i) > 0$ and in some cases can be larger than the CI indices of the other nodes. So the CI algorithm may say node i is more important to remove first, as its removal decreases mostly the eigenvalue of the non-backtracking matrix. After a moment of thought we see that this conclusion is not correct, as removing node i does not change the eigenvalue of the non-backtracking matrix at all, because the eigenvalue of the non-backtracking matrix is the same as the 2-core of the network, while node i does not belong to the 2-core of the network.

Appendix C: Comparing CoreHD and CoreCI

Since performing node deletion on the network 2-core is the key of CoreHD’s good performance, it is natural to expect that the CI algorithm can also be improved by adding the 2-core reduction process. To confirm this, we implement an extended CI algorithm (named as CoreCI) as follows. At each elementary node removal step, (1) the 2-core of the remaining network is obtained by cutting leaves recursively as in CoreHD, and then (2) the CI index of each node in the 2-core is computed by considering only nodes and links within this 2-core, and finally (3) a node with the highest CI index is deleted from the 2-core. Similar to CoreHD and BPD, after a forest is produced by CoreCI, we then perform a greedy tree-breaking process if necessary and then re-insert some nodes back to the network as long as the size of the largest connected component is still below the threshold value of (say) $0.01N$.

We indeed observe that CoreCI performs considerably better than the original CI algorithm. However it does not outperform CoreHD. We list in Table II the comparative results of CoreHD versus CoreCI on ER, RR, and SF random networks. Notice that the fractions ρ of deleted nodes by CoreHD and CoreCI are very close to each other, with CoreHD performs slightly better. These results clearly demonstrate that the CI index is not a better indicator of node importance than the degree in the 2-core. Because repeatedly computing the CI indices within the 2-core is still very time-consuming, we recommend CoreHD rather than CoreCI as an efficient heuristic for practical applications.

TABLE II. Comparing the dismantling performance of CoreHD and CoreCI on ER, RR, and SF random networks. Each data point is the mean and standard deviation of ρ (the fraction of deleted nodes) over 96 dismantling solutions obtained by CoreHD or CoreCI on 96 independent network instances of size $N = 10^5$ and mean degree c (ER and SF) or degree K (RR). The ball radius of CoreCI is fixed to $\ell = 4$. The SF network instances are generated by the static method [18].

ER		RR		SF ($\gamma = 3.0$)				
c	CoreHD	CoreCI	K	CoreHD	CoreCI	c	CoreHD	CoreCI
3.0	0.1413(3)	0.1427(3)	3	0.25043(3)	0.2539(2)	3.0	0.0886(3)	0.0893(3)
4.0	0.2226(4)	0.2249(4)	4	0.3464(2)	0.3564(3)	4.0	0.1373(4)	0.1383(4)
5.0	0.2908(4)	0.2937(4)	5	0.4110(2)	0.4239(3)	5.0	0.1820(5)	0.1833(5)
6.0	0.3476(4)	0.3509(4)	6	0.4605(3)	0.4733(3)	6.0	0.2222(5)	0.2237(5)
7.0	0.3954(4)	0.3990(4)	7	0.5004(3)	0.5128(3)	7.0	0.2582(5)	0.2560(5)
8.0	0.4361(4)	0.4400(5)	8	0.5335(3)	0.5455(3)	8.0	0.2906(6)	0.2925(6)
9.0	0.4712(4)	0.4752(5)	9	0.5617(3)	0.5733(3)	9.0	0.3196(5)	0.3217(5)
10.0	0.5018(4)	0.5060(4)	10	0.5861(3)	0.5974(4)	10.0	0.3460(6)	0.3481(6)
11.0	0.5288(4)	0.5330(4)	11	0.6075(3)	0.6182(4)	11.0	0.3699(6)	0.3723(6)
12.0	0.5527(4)	0.5571(4)	12	0.6264(3)	0.6367(3)	12.0	0.3918(6)	0.3943(6)